# Numbers

* **Set:** a collection of objects called **elements**.
* We write x ∈ S to mean that element x is in set S.
* A set is **nonempty** if it has at least one element.
* The empty set is denoted by ∅
* **Subset:** A subset of S is a set T with the property .
* Every element of T is an element of S.
* Trivially, S ⊆ S and ∅ ⊆ S
* The subset symbol is denoted by ⊆.
* The set of Natural Numbers ℕ = {1, 2, 3, …} is useful for counting and for ordering.
* The order symbols are <, ≤, ≥, >

## Set Algebra

* An **operation** on a set S is a rule for combining elements of S.
* **Binary operations:** combines pairs of elements to prove another.

A binary operation ∗ is **closed** if:

|  |
| --- |
| Definition |
|  |

* Four common operations on numbers are .

Exercise:

Are +, -, ⋅, / closed on ℕ? Prove or disprove



An element e ∈ S is called an **identity** if:

|  |
| --- |
| Definition |
|  |

Exercise:

Does ℕ have an identity under +? Under ⋅?



If identity of S, an element is called **invertible** when :

|  |
| --- |
| Definition |
|  |

Then **y** is called the **inverse** of **x**.

Exercise:

What are the invertible elements of ℕ under +, ⋅?



A binary operation ∗ on S is **commutative** if:

|  |
| --- |
| Definition |
|  |

It is **associative** if:

|  |
| --- |
| Definition |
|  |

* The operations +, ⋅ are associative and commutative on ℕ.

Exercise:

Rock-Paper-Scissors.

Let M = {r, p, s} and consider the binary operation that gives the winner of the game.



Is ∗ associative?



A binary operation ∗ is **distributive** over another ∙ if for all .

|  |
| --- |
| Definition |
|  |

For example, multiplication distributes over addition on ℕ.

Exercise:

Prove that addition does not distribute over multiplication on ℕ.



Exercise:

Let a, b ∈ ℕ. Simplify the following expression, giving reasons for each step. [8(a + b)] + 2a



A set S with order ≤ is called **well-ordered** if every nonempty subset T of S has at least one smallest element.

|  |
| --- |
| Definition |
| That is, if , then |

The set ℕ with the usual order ≤ is well-ordered.

The set of integers ℤ = {…, -2, -1, 0, 1, 2, …} can be constructed from ℕ:

* It is the set of differences {m-n} ∀ m, n ∈ ℕ.
* The order ≤ on ℕ extends to ℤ.

Exercise:

1. Are +, -, ⋅, / closed on ℤ?
2. Does ℤ has identities under +, ⋅?
3. What are the invertible elements of ℤ under +, ⋅?



* On ℤ, + and ⋅ are commutative and associative.
* On ℤ, - and / are **not** commutative and associative.
* However, if we define a **– b = a + (-b)** and **a/b = a ⋅ 1/b**, then we have commutativity and associativity.

(associativity)

(distribution)

* Multiplication distributes over addition and subtraction on ℤ:

Exercise:  
Is ℤ well-ordered?



## Some Common Rules

An integer is **even** if for some .

An integer is **odd** if for some

An integer is **prime** if whenever for , either or

An integer is **composite** if it is not prime (i.e. with )

* The set of Rationals ℚ is the set of numbers that can be written
* ℚ can be constructed from ℤ.

## Dedekind Cuts

* To construct the Real Numbers ℝ, we can use ℚ and the Dedekind Cuts.
* A Dedekind Cut of ℚ is a pair of subsets of ℚ that satisfy the following:
* and are nonempty
* is closed downwards: If and , then
* is closed upwards: if and , then
* contains no greatest element:
* Given , we can form a Dedekind Cut (A,B) where:

AND

* That is the Dedekind-Cut identification of all rational numbers
* But we can make such cuts at non-rational numbers as well.
* An irrational number is one that cannot be written as .
* An example is

Exercise:

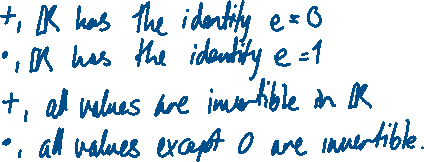
Prove that



* The following Dedekind Cut defines :
* The numbers defined by ALL Dedekind Cuts of ℚ make up the set of Real Numbers ℝ.
* The usual order ≤ on ℝ is inherited from ℕ.

Exercise:

1. Which of are closed on ℝ?
2. Does ℝ have identities under ?
3. What are the invertible elements of ℝ under ?



* As in ℚ, the operations , are commutative and associative.
* In ℝ, , are not commutative and associative, unless you define them as we did in ℚ.